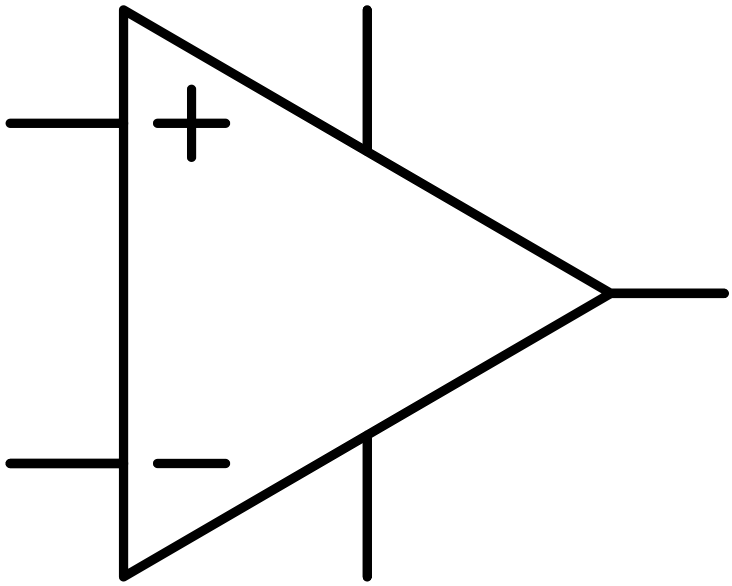
Fun with op-amp band-pass filters



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Two nice op-amp band-pass filter circuits came up to my knowledge recently and since I spent a decent amount of time deriving equations for those circuits it is only fair to write the equations down and share them. Both of the circuits relate to equaliser applications, although a complete graphic equaliser is not presented in this context.

1 Adjustable single op-amp band-pass booster

Figure 1 shows a circuit where a bridged-tee filter section is connected in the op-amp feedback loop. Because the bridged-tee configuration contains two capacitors, the resulting transfer function will be a second-order (biquad) filter function. The bridged-tee circuit implements a notch filter on its own, but when connected in the feedback loop of an active circuit (op-amp or transistor), the functionality is inverted into a band-pass filter. The band-pass output is obtained from the op-amp output pin, labelled as 4 in Figure 1.

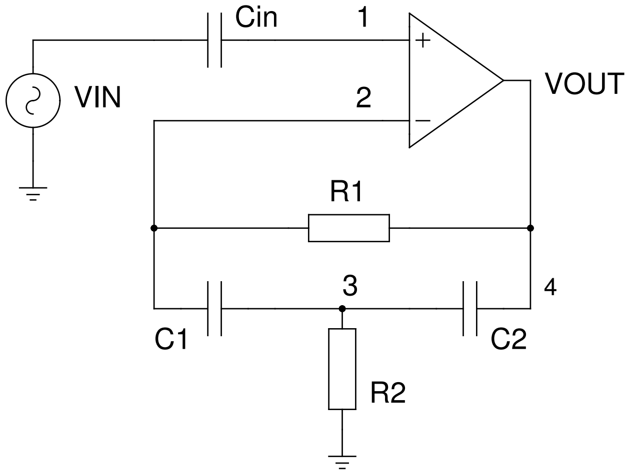

Figure 1: Bridged-tee network realises an op-amp biquad filter

When adding a variable resistor beside the resistor R_1 in the circuit of Figure 1, the gain and the quality factor Q of the filter can be adjusted. The circuit can be then used for boosting a specified frequency range and using several of these filters in series, one has a relatively simple equaliser circuit for some selected frequencies.

But before analysing the circuit using the variable resistor for gain control, let's write the equations for the circuit shown in Figure 1. The nodal-analysis matrix equation describing the circuit is written as:

$$\begin{bmatrix} sC_{\text{in}} & 0 & 0 & 0 \\ 0 & \dfrac{1}{R_1} + sC_1 & -sC_1 & -\dfrac{1}{R_1} \\ 0 & -sC_1 & \dfrac{1}{R_2} + sC_1 + sC_2 & -sC_2 \\ 0 & -\dfrac{1}{R_1} & -sC_2 & \dfrac{1}{R_1} + sC_2 \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} sC_{\text{in}} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since ideal op-amps have the property that the voltage of both inputs are equal, one can simplify the matrix equation. After adding terms of V_1 into V_2 and removing the last row of the matrix equation, one is left with an equation:

$$\begin{bmatrix} sC_{\text{in}} & 0 & 0 \\ \dfrac{1}{R_1} + sC_1 & -sC_1 & -\dfrac{1}{R_1} \\ -sC_1 & \dfrac{1}{R_2} + sC_1 + sC_2 & -sC_2 \end{bmatrix} \times \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} sC_{\text{in}} \\ 0 \\ 0 \end{bmatrix}.$$

Solving for the node voltage V_4, which is the output voltage V_{out}, yields a transfer function

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{s^2 C_1 C_2 R_1 R_2 + sC_1 R_1 + sC_2 R_2 + sC_1 R_2 + 1}{s^2 C_1 C_2 R_1 R_2 + sC_2 R_2 + sC_1 R_2 + 1} \tag{1}$$

It is easy to notice that the numerator and the denominator are very similar. The transfer function can be simplified further, but let's not do that quite yet. Before that, let's consider the slightly modified circuit of Figure 2, which allows to make the gain adjustable by a potentiometer.

The transfer function of this circuit is easily obtained from the simpler model by noticing that R_x is in series with C_1 and there is not any new circuit node in between these two components. In this case we can evaluate the combined impedance of R_x and C_1 which is:

$$Z = \frac{1 + sC_1 R_x}{sC_1},$$

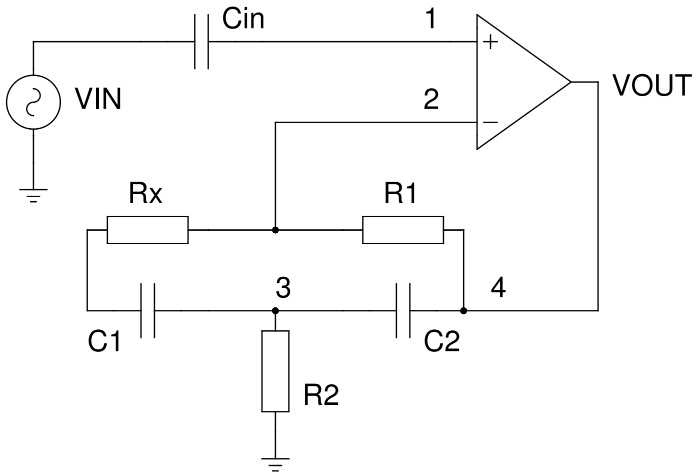

Figure 2: Gain-controllable band-pass filter

the admittance of this is

$$Y = \frac{sC_1}{1 + sC_1R_x},$$

which can be inserted in equation (1) in place of C_1 to yield

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{s^2C_1C_2R_2(R_1 + R_x) + sC_2R_2 + sC_1R_2 + sC_1(R_1 + R_x) + 1}{s^2C_1C_2R_2(R_1 + R_x) + sC_2R_2 + sC_1R_2 + sC_1R_x + 1} \tag{2}$$

After simplification this transfer function starts to resemble the gain expression of the non-inverting op-amp configuration $(1 + G)$:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = 1 + \frac{s\dfrac{R_1}{C_2R_2(R_1 + R_x)}}{s^2 + s\dfrac{1}{R_1 + R_x}\left[\dfrac{1}{C_1} + \dfrac{1}{C_2}\left(1 + \dfrac{R_x}{R_2}\right)\right] + \dfrac{1}{C_1C_2R_2(R_1 + R_x)}}$$

From the denominator expression one can evaluate the centre-frequency f_c and the quality factor Q, which are

$$f_c = \frac{1}{2\pi\sqrt{C_1C_2R_2\left(R_1 + R_x\right)}}$$

and

$$Q = \frac{\sqrt{\dfrac{R_1 + R_x}{R_2}}}{\sqrt{\dfrac{C_2}{C_1}} + \sqrt{\dfrac{C_1}{C_2}}\left(1 + \dfrac{R_x}{R_2}\right)}.$$

In addition, if the input signal is fed through the combination of a high-pass filter of C_{in} and R_{in}, the effect of this can be extracted as a separate multiplier,

$$\text{Factor from input filter: } \frac{1}{1+\dfrac{1}{sC_{\text{in}}R_{\text{in}}}}\frac{V_{\text{out}}}{V_{\text{in}}} \qquad \rightarrow \qquad 2\pi C_{\text{in}}R_{\text{in}} > 1.$$

The condition on the right side states that if the time constant of the input filter is large enough (larger than 1, for example), the input filter does not affect the output of the band-pass filter.

Figure 3 shows a quick example on the output of the band-pass filter when using the potentiometer to control the gain of the filter. The gain scale has been transformed to show decibels. The component values used are: $C_1 = 150$ nF, $C_2 = 10$ nF, $R_2 = 22$ k, $R_1 = 47 + aR_v$ k and $R_v = (1-a)50$ k. Here a can have values from $[0:1]$ and therefore is used for representing the turn-percentage of the variable resistor. The idea is that the other portion of the pot is added into the value of R_1. This is because of the analysis method used.

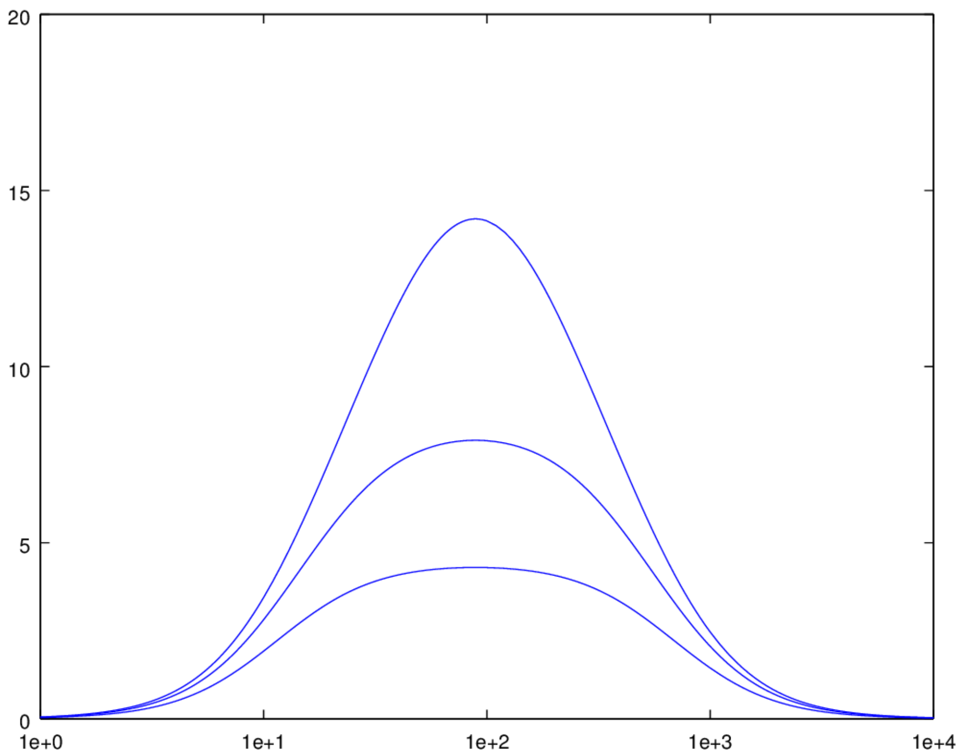

Figure 3: Quick simulation presentation with different control-pot values

2 State variable filter based frequency band control

First we will show the basic state variable filter, which is typically found from literature as drawn in Figure 4. The reader should note that in this version the input signal is fed into the inverting pin of the operational amplifier.

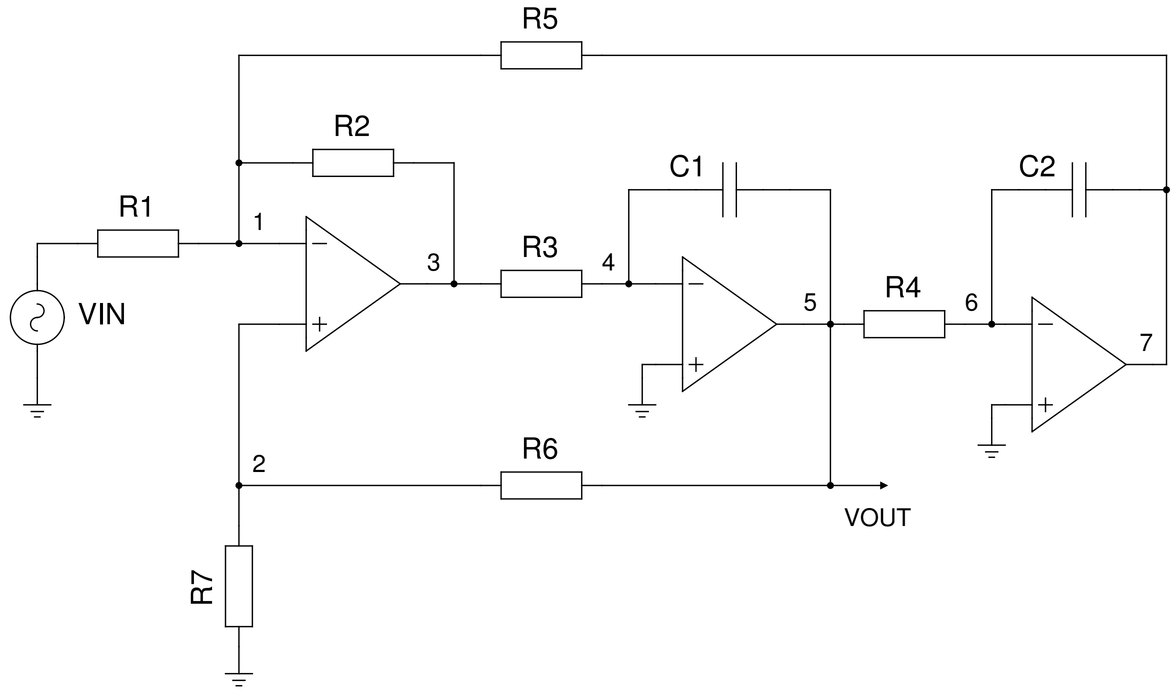

Figure 4: State variable filter, input on inverting pin, band-pass output

So, without further ado, let's analyse the circuit using the nodal matrix method for ideal operational amplifiers. In this method, first the nodal admittance matrix is written normally and then it is reduced based on the fact that the input nodes (plus and minus) of an op-amp are virtually in same potential and that the output current can be left out to be solved later.

The nodal matrix equation for the circuit of Figure 4 is:

$$\begin{bmatrix} Y_{11} & 0 & -Y_{13} & 0 & 0 & 0 & -Y_{17} \\ 0 & Y_{22} & 0 & 0 & -Y_{25} & 0 & 0 \\ -Y_{13} & 0 & Y_{33} & -Y_{34} & 0 & 0 & 0 \\ 0 & 0 & -Y_{43} & Y_{44} & -Y_{45} & 0 & 0 \\ 0 & -Y_{52} & 0 & -Y_{54} & Y_{55} & -Y_{56} & 0 \\ 0 & 0 & 0 & 0 & -Y_{65} & Y_{66} & -Y_{67} \\ -Y_{71} & 0 & 0 & 0 & 0 & -Y_{76} & Y_{77} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} \frac{V_{\text{in}}}{R_1} \\ 0 \\ I_1 \\ 0 \\ I_2 \\ 0 \\ I_3 \end{bmatrix}.$$

The nonzero elements $Y_{i,j}$ of the admittance matrix are marked with their corresponding indices indicating the row and column. The listing (3) contains the actual terms that should be substituted to the admittance matrix above.

$$
\begin{aligned}
Y_{11} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5} & Y_{13} &= Y_{31} = \frac{1}{R_2} \\
Y_{22} &= \frac{1}{R_6} + \frac{1}{R_7} & Y_{25} &= Y_{52} = \frac{1}{R_6} \\
Y_{33} &= \frac{1}{R_2} + \frac{1}{R_3} & Y_{34} &= Y_{43} = \frac{1}{R_3} \\
Y_{44} &= \frac{1}{R_3} + sC_1 & Y_{45} &= Y_{54} = sC_1 \\
Y_{55} &= \frac{1}{R_4} + sC_1 & Y_{56} &= Y_{65} = \frac{1}{R_4} \\
Y_{66} &= \frac{1}{R_4} + sC_2 & Y_{67} &= Y_{76} = sC_2 \\
Y_{77} &= \frac{1}{R_5} + sC_2 & Y_{71} &= Y_{17} = \frac{1}{R_5}
\end{aligned}
\tag{3}
$$

In the case of ideal operational amplifiers, the node voltages V_1 and V_2 are at the same potential, and therefore column 2 can be added to column 1 and column 2 is removed. Also, because V_4 and V_6 are grounded, the columns multiplying these voltages can be removed. To reshape the matrix into a square, all the rows that are aligned with the currents I_1, I_2 and I_3 are removed. With these reductions, the 7x7 matrix is now a 4x4 matrix:

$$
\begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5} & -\frac{1}{R_2} & 0 & -\frac{1}{R_5} \\
\frac{1}{R_6} + \frac{1}{R_7} & 0 & -\frac{1}{R_6} & 0 \\
0 & -\frac{1}{R_3} & -sC_1 & 0 \\
0 & 0 & -\frac{1}{R_4} & -sC_2
\end{bmatrix}
\times
\begin{bmatrix} V_1 \\ V_3 \\ V_5 \\ V_7 \end{bmatrix}
=
\begin{bmatrix} \frac{V_{\text{in}}}{R_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

From this matrix equation, one can solve the transfer functions for every op-amp output node using the Cramer's rule. The rule leads to an equation having determinants in the numerator and in the denominator. Each output node of this circuit will realise a different filter function.

For this basic state variable filter it is well known that the band-pass output is obtained from node 5. So, after solving the determinant quotient for node

5, the band-pass transfer function is

$$\frac{V_5}{V_{\text{in}}} = \frac{s\dfrac{1}{C_1R_3}\dfrac{R_2}{R_1}}{s^2 + s\dfrac{R_7(R_2R_5 + R_1R_5 + R_1R_2)}{C_1R_1R_3R_5(R_6 + R_7)} + \dfrac{1}{C_1C_2R_3R_4}\dfrac{R_2}{R_5}}$$

This is standard textbook stuff.

Next we will analyse the same state variable filter, but the input signal is taken into the non-inverting pin of the op-amp. This configuration is shown in Figure 5. Surprisingly this configuration is rarely referenced in related literature, even though in many cases this configuration would be more simpler to adapt to different designs. The following analysis shows why this configuration is so 'elegant'.

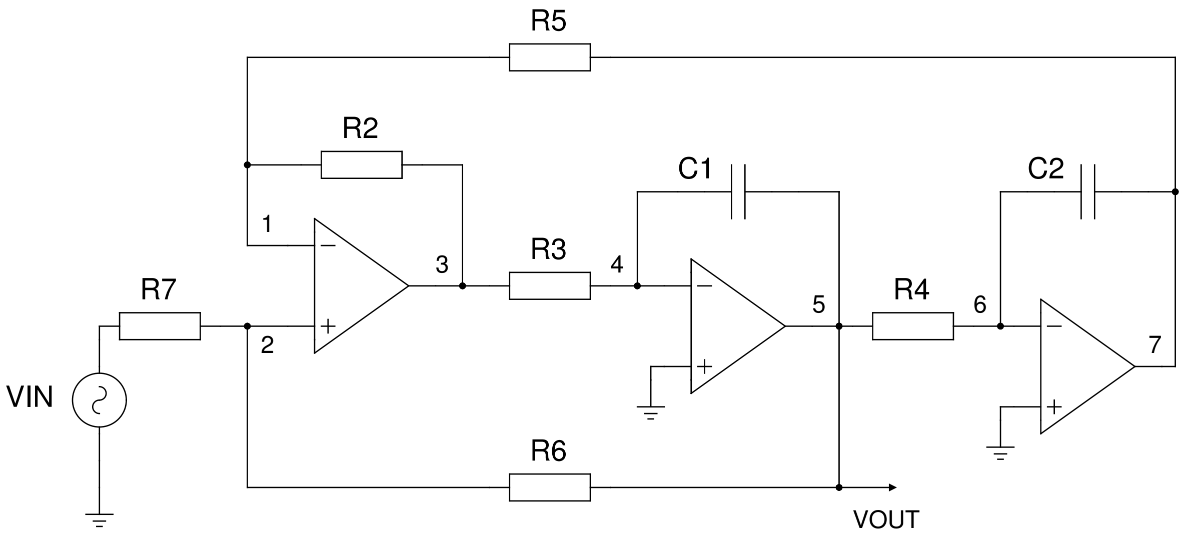

Figure 5: State variable filter, input on non-inverting pin, band-pass output

The reduced matrix for the circuit of Figure 5 is the same as for the inverting-input filter, except that R_1 is taken out, and the input current source $\frac{V_{\text{in}}}{R_7}$ is attached to node 2 instead of node 1. With these changes, the matrix equation is

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_5} & -\frac{1}{R_2} & 0 & -\frac{1}{R_5} \\ \frac{1}{R_6}+\frac{1}{R_7} & 0 & -\frac{1}{R_6} & 0 \\ 0 & -\frac{1}{R_3} & -sC_1 & 0 \\ 0 & 0 & -\frac{1}{R_4} & -sC_2 \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_3 \\ V_5 \\ V_7 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{V_{\text{in}}}{R_7} \\ 0 \\ 0 \end{bmatrix}.$$

and the band-pass filtered output obtained from node 5 is now

$$\frac{V_5}{V_{\text{in}}} = \frac{-s\frac{1}{C_1R_3}\frac{R_6}{R_5}\left(\frac{R_2+R_5}{R_6+R_7}\right)}{s^2+s\frac{1}{C_1R_3}\frac{R_7}{R_5}\left(\frac{R_2+R_5}{R_6+R_7}\right)+\frac{1}{C_1C_2R_3R_4}\frac{R_2}{R_5}} \tag{4}$$

And hey, this looks much simpler! Also note that in this case the output signal is inverted from the original input signal. Another important observation for future purposes.

The general form of the band-pass transfer function is written as

$$T(s) = \frac{a_1 s}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}, \tag{5}$$

and the centre-frequency gain is then

$$\frac{a_1 Q}{\omega_0}. \tag{6}$$

Using this gain information on the band-pass equation (4), we notice that the gain of this configuration is defined simply by $\frac{R_6}{R_7}$. It cannot get any simpler than this!

Now we are ready to check out the complete band-pass control circuit, based on the state variable filter of Figure 5. Here, by common equaliser theory, one needs to choose unity gain for the state variable filter, so here $R_6 = R_7$.

The added circuit is a standard difference amplifier, where the variable resistor mixes the input and output together, and sends the result of mixing into the state variable filter. The resistor R_x is there to add gain, if it is left out, the gain at the pass-band will be ± 9 dB. Meaning that this circuit does both band-pass (boost) and notch (cut) filtering, depending on the setting on the variable resistor. It is beneficial to analyse the two situations where the pot is at the left extreme position and at the right extreme position.

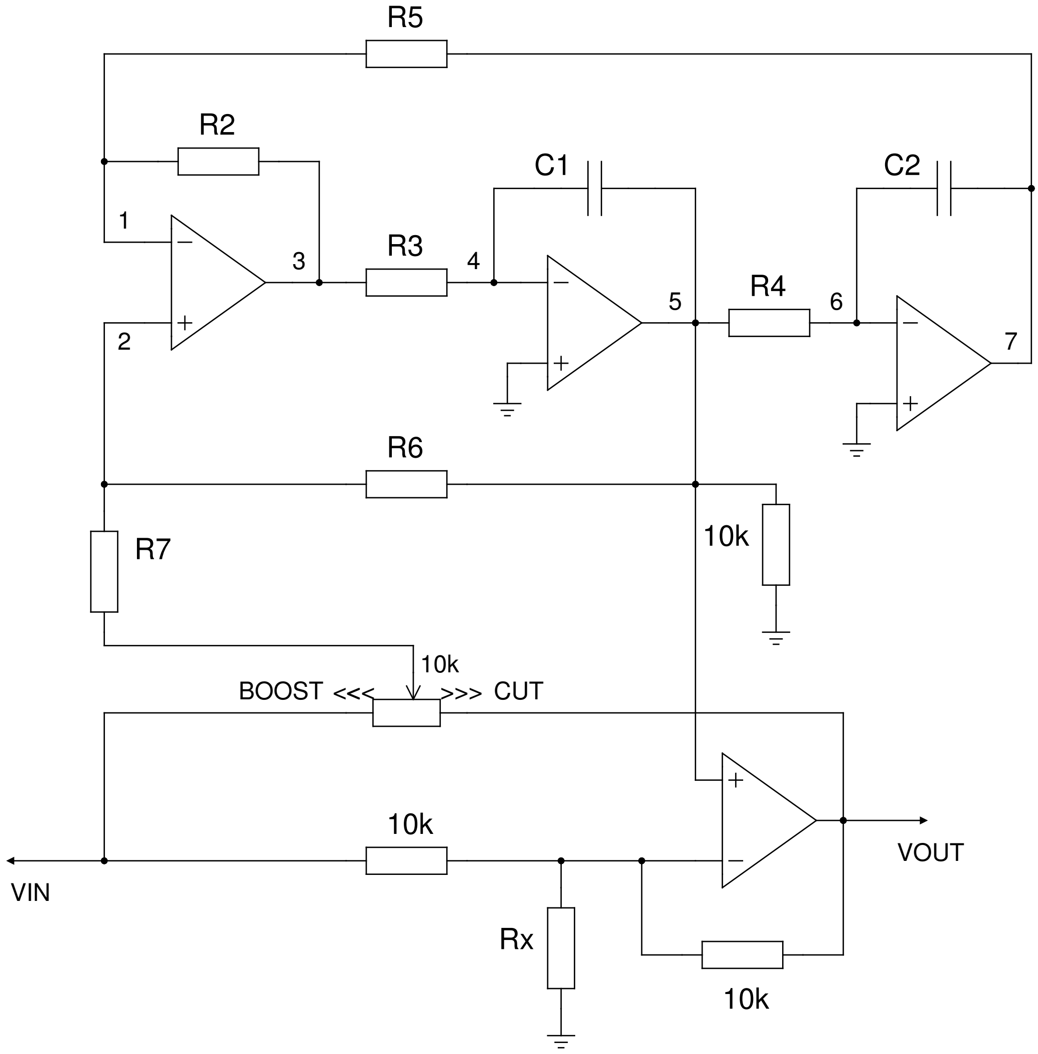

Figure 6: Gain-controllable band-pass using the unity-gain state-variable filter

To prove that the gain for the output signal is defined only by the properties of the difference amplifier, the difference amplifier is analysed separately. The state variable filter is taken along only by taking the output impedance of the filter as one of the resistors in the difference amplifier.

First we will analyse the basic version, without the gain resistor R_x. This basic difference amplifier is shown in Figure 7.

For the circuit in Figure 7, the nodal equations for the boost mode, where the input voltage finds its way inverted through the filter to the V_+ pin are:

$$\frac{V_- - V_{\text{in}}}{R_1} + \frac{V_- - V_{\text{out}}}{R_2} = 0$$

$$\frac{V_+ - (-V_{\text{in}})}{R_{\text{svf}}} + \frac{V_+}{R_4} = 0.$$

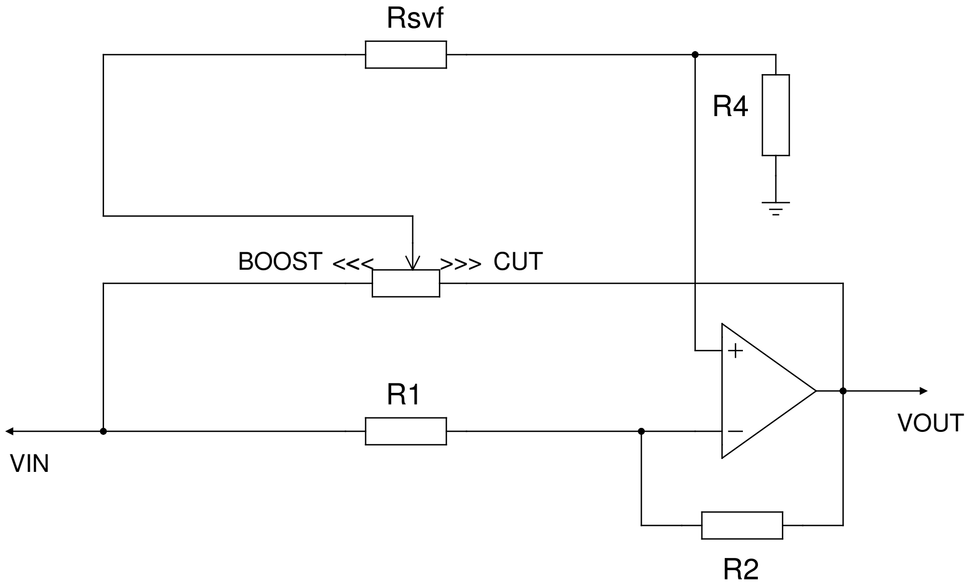

Figure 7: Difference amplifier having the output impedance of the state variable filter as R_{svf}. The mixing resistor does not affect the difference calculation.

The inversion of the input signal at V_+ should be clear from the equations above when using the double negation sign. By default we are assuming that all the currents are approaching the node, therefore the currents are added together using the $+$-sign.

To solve the equation pair, we proceed as follows. Because in an ideal op-amp the $+$ and $-$ pins are in the same potential, one can solve for these two voltages

$$V_- = \frac{V_{\text{in}} R_2 + V_{\text{out}} R_1}{R_1 + R_2}$$

$$V_+ = \frac{-V_{\text{in}} R_4}{R_{\text{svf}} + R_4} \approx -V_{\text{in}}.$$

The approximation can be made because the output impedance of the state variable filter is very small compared to R_4. The solved input voltages can now be assigned equal. This will give an equation

$$-V_{\text{in}} = \frac{V_{\text{in}} R_2 + V_{\text{out}} R_1}{R_1 + R_2}, \tag{7}$$

which results in a transfer function

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\left(1 + 2\frac{R_2}{R_1}\right), \tag{8}$$

for the boost mode.

Nodal equations for the cut mode are:

$$\frac{V_- - V_{\text{in}}}{R_1} + \frac{V_- - V_{\text{out}}}{R_2} = 0$$

$$\frac{V_+ - (-V_{\text{out}})}{R_{\text{svf}}} + \frac{V_+}{R_4} = 0.$$

In the cut mode, the output signal is fed through the state variable filter to the $+$ pin of the difference amplifier. The state variable filter inverts the output signal, which is taken into account by placing the extra minus sign in to the latter equation, where the output voltage is involved.

The solution procedure is the same as already used before. The voltages at the difference amplifier inputs are solved as

$$V_- = \frac{V_{\text{in}} R_2 + V_{\text{out}} R_1}{R_1 + R_2}$$

$$V_+ = \frac{-V_{\text{out}} R_4}{R_{\text{svf}} + R_4} \approx -V_{\text{out}}.$$

After combining these equations, we will have an equation

$$-V_{\text{out}} = \frac{V_{\text{in}} R_2 + V_{\text{out}} R_1}{R_1 + R_2}, \tag{9}$$

which results in a transfer function

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{1}{1 + 2\dfrac{R_1}{R_2}} \tag{10}$$

for the cut mode.

Hence, if it is chosen that $R_1 = R_2$, then the circuit combination of the unity gain state variable filter and the difference amplifier will give a boost gain of -3 and a cut gain of $-\frac{1}{3}$. In decibels these gain values are approximately $+9$ dB and -9 dB respectively.

To summarise the results, the transfer functions for boost and cut are:

$$\text{BOOST: } \frac{V_{\text{out}}}{V_{\text{in}}} = -\left(1 + 2\frac{R_2}{R_1}\right) \quad ; \quad \text{CUT: } \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{1}{1 + 2\dfrac{R_1}{R_2}} \tag{11}$$

Figure 8 shows the same difference amplifier, but with added resistor R_x. Soon we will find out what is the reason for adding the mysterious extra resistor. The nodal equations for boost and cut in this case are

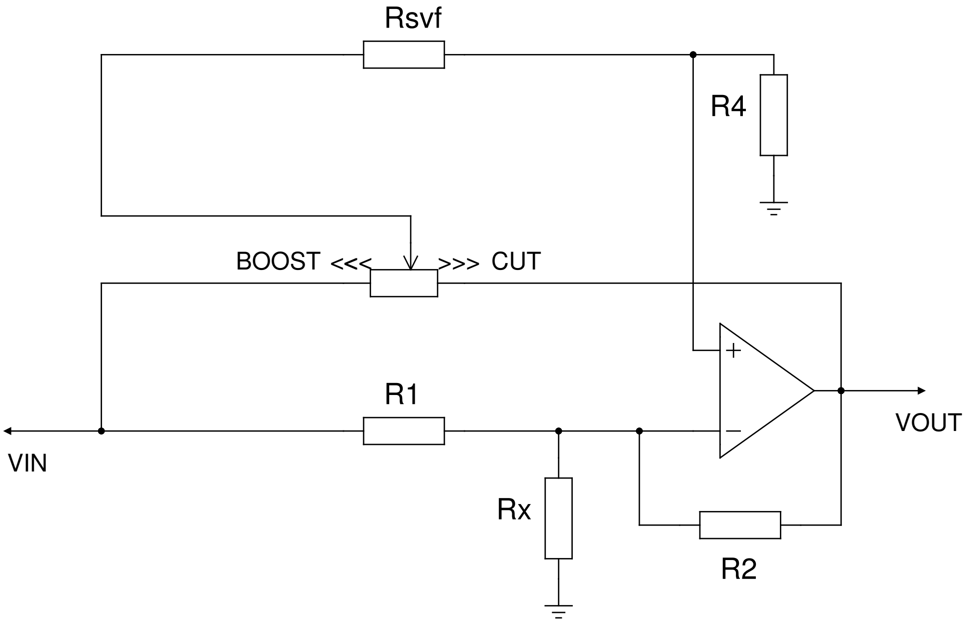

Figure 8: Difference amplifier having the output impedance of the state variable filter as R_{svf} and with extra R_x for determining the maximum gain

$$\frac{V_- - V_{\text{in}}}{R_1} + \frac{V_- - V_{\text{out}}}{R_2} + \frac{V_-}{R_x} = 0$$

$$\frac{V_+ - (-V_{\text{in}})}{R_{\text{svf}}} + \frac{V_+}{R_4} = 0,$$

for boost and

$$\frac{V_- - V_{\text{in}}}{R_1} + \frac{V_- - V_{\text{out}}}{R_2} + \frac{V_-}{R_x} = 0$$

$$\frac{V_+ - (-V_{\text{out}})}{R_{\text{svf}}} + \frac{V_+}{R_4} = 0,$$

for cut. These are almost the same as the previous equations, only the additional term with R_x is introduced. The reader is encouraged to verify that the transfer functions resulting from these equations are:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\left(1 + 2\frac{R_2}{R_1} + \frac{R_2}{R_x}\right) \tag{12}$$

for boost, and

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{1}{1 + 2\frac{R_1}{R_2} + \frac{R_1}{R_x}} \tag{13}$$

for cut. The obvious conclusion is that if R_x is chosen to be smaller than R_1 and/or R_2, then the gain is increased from the default 9 dB amplification.

Next for some fancy visualisation of the results. Figure 9 presents the output frequency response of both boost and cut modes. Both modes are obtained from the same circuit only by adjusting the variable resistor either in left (for boost) or right (for cut) extreme position. The middle position of the variable resistor should give 0 dB gain for all frequencies, thereby being an all-pass filter.

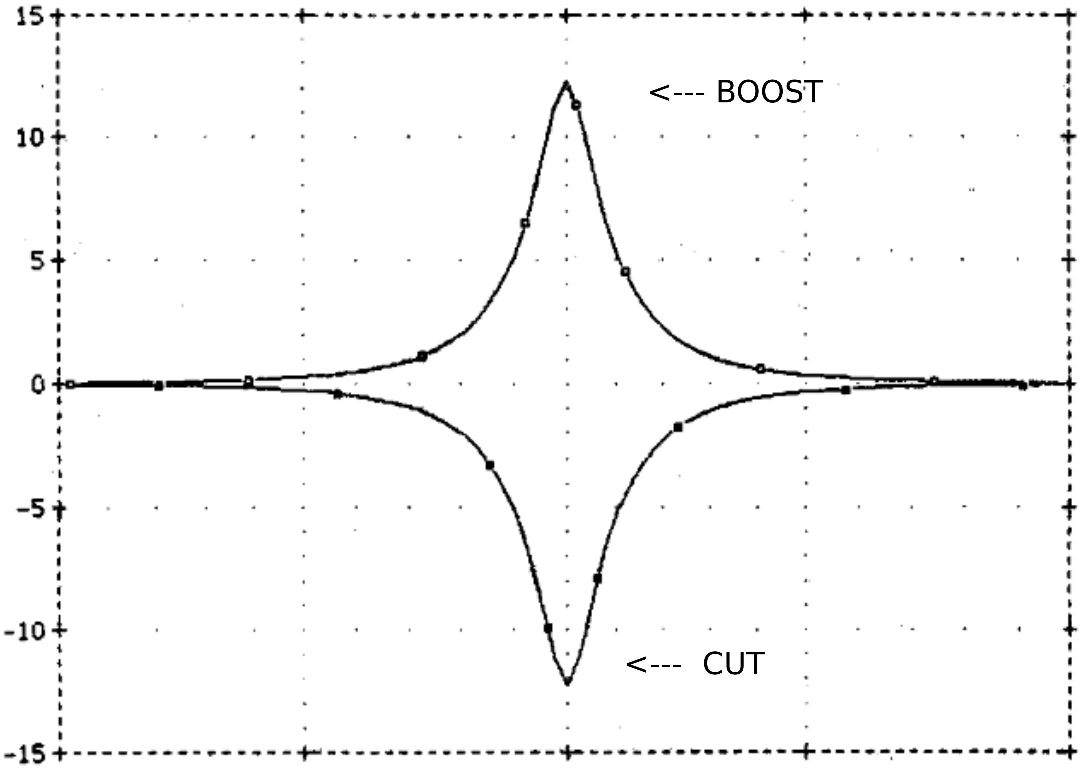

Figure 9: Maximum boost and cut output from the state variable filter + difference amplifier. Intermediate values are obtained by setting the variable resistor between the two extremes. All-pass filter state should be possible when pot at mid-position. This image is copied from RANE-article 'Constant-Q Graphic Equalizers' by Dennis A. Bohn. Sorry for breaking the copyright laws only because of being lazy.

Don't know if this was that much fun after all ... But hopefully useful.