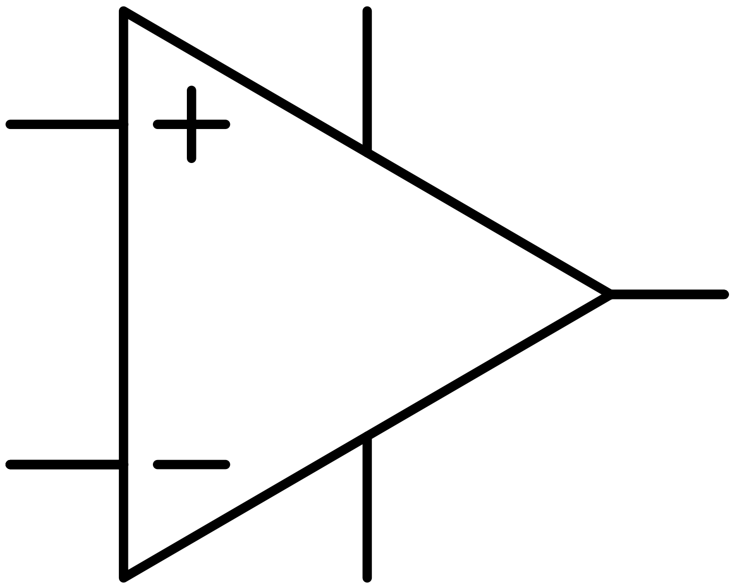
Calculating the voltage gain and output impedance of operational amplifier stages



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This example shows how to evaluate the small-signal voltage gain and the output impedance of the inverting and non-inverting operational amplifier (op-amp) stages. The output impedance will be evaluated using the results of Thévenin and Norton theorems. According to these theorems, the output impedance of any circuit stage is obtained as the quotient of open circuit voltage and short circuit current. The following analysis demonstrates how to apply both nodal and mesh analysis methods to op-amp circuits.

1 The inverting operational amplifier

Figure 1 shows the inverting op-amp configuration with resistors R_1 and R_2, which form an external feedback loop across the op-amp. If the operational amplifier component is assumed to be ideal (infinite internal gain and infinite input resistance), the resistors R_1 and R_2 completely define the small-signal gain of the inverting configuration. The following derivation indicates the mathematical gain formulae in the case where the op-amp has a finite internal gain and input resistance.

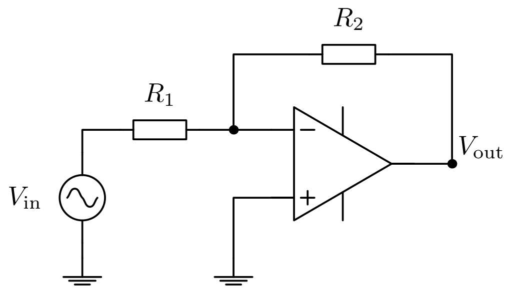

Figure 1: The inverting operational amplifier configuration

Prior to determining the output impedance of the inverting op-amp configuration it is necessary to calculate the voltage gain of the amplifier stage. Since voltages are the object of interest, the nodal analysis method is suitable for determining the relationship between the output and input voltages. The nodal analysis requires the signal sources to be represented as current sources. Due to this, the small-signal equivalent circuit needs to be redrawn after applying the required source transformations. Figure 2 shows the small-signal model, from where the voltage gain of the inverting op-amp configuration can be evaluated with nodal analysis.

After the input voltage source and the voltage-controlled voltage source have

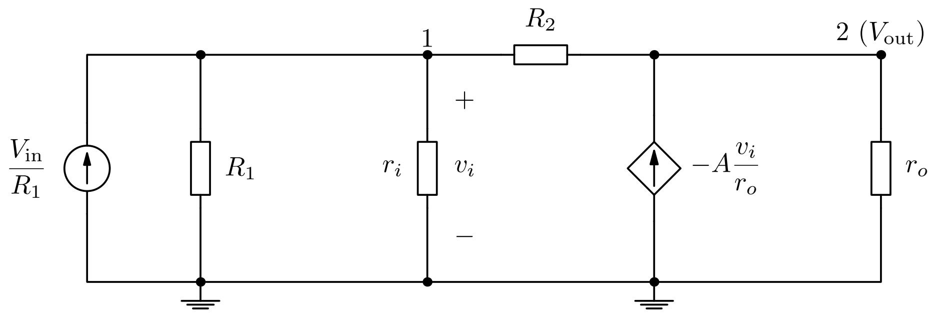

Figure 2: The small-signal model of the inverting operational amplifier

been converted into current sources with resistor R_1 and internal output resistance r_o respectively, the small-signal model is represented by the matrix equation

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{r_i} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{r_o} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{V_{\text{in}}}{R_1} \\ -\frac{Av_i}{r_o} \end{bmatrix}.$$

The gain term $-\frac{Av_i}{r_o}$ still needs to be transferred from the current vector to the admittance matrix. Because $v_i = V_1$, the final form of the matrix equation is

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{r_i} & -\frac{1}{R_2} \\ -\frac{1}{R_2} + \frac{A}{r_o} & \frac{1}{R_2} + \frac{1}{r_o} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{V_{\text{in}}}{R_1} \\ 0 \end{bmatrix}.$$

Solving for the node voltage V_2, which is the output voltage V_{out}, yields a fractional gain term

$$V_{\text{out}} = V_2 = \frac{V_{\text{in}}\left(r_o - AR_2\right)}{R_1\left(1 + A\right) + r_o + R_2 + \frac{r_o R_1}{r_i} + \frac{R_1 R_2}{r_i}}.$$

This result is used later together with the expression of the short circuit current to solve a formula for the output impedance of the inverting op-amp configuration.

Another small-signal model is needed to determine the short circuit current using the mesh analysis. Figure 3 indicates the chosen current loops and their directions inside the circuit branches.

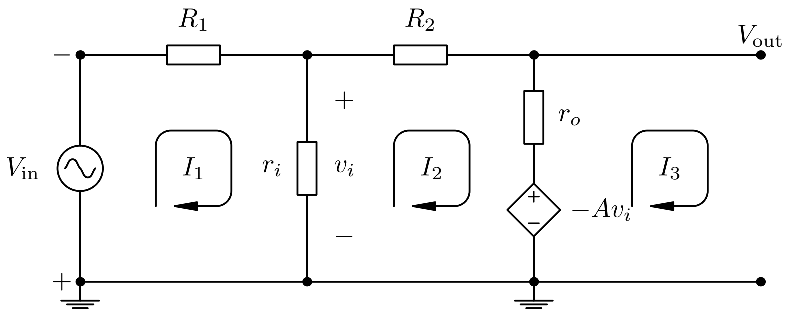

Figure 3: The small-signal model of the inverting operational amplifier

The short circuit current is evaluated from the matrix equation

$$\begin{bmatrix} r_i + R_1 & -r_i & 0 \\ -r_i & r_i + R_2 + r_o & -r_o \\ 0 & -r_o & r_o \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} \\ Av_i \\ -Av_i \end{bmatrix}.$$

The voltage v_i can be expressed using the mesh currents as $v_i = I_2 R_2$ because the other end of R_2 is shorted to the ground. Therefore, the matrix can be reshaped to

$$\begin{bmatrix} r_i + R_1 & -r_i & 0 \\ -r_i & r_i + R_2(1-A) + r_o & -r_o \\ 0 & -r_o + AR_2 & r_o \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} \\ 0 \\ 0 \end{bmatrix}.$$

The solution of this matrix equation with respect to mesh current I_3 is

$$I_3 = \frac{V_{\text{in}}\left(r_o - AR_2\right)}{r_o\left(R_1 + R_2\right) + R_1 R_2 \dfrac{r_o}{r_i}}.$$

The output impedance of the inverting configuration is therefore

$$Z_{\text{out}} = \frac{V_2}{I_3} = \frac{r_o\left(R_1 + R_2\right) + R_1 R_2 \dfrac{r_o}{r_i}}{R_1\left(1 + A\right) + r_o + R_2 + \dfrac{r_o R_1}{r_i} + \dfrac{R_1 R_2}{r_i}}.$$

2 Non-inverting operational amplifier

Figure 4 shows the non-inverting op-amp configuration with resistors R_1 and R_2, which form an external feedback loop across the op-amp. If the operational amplifier component is assumed to be ideal (infinite internal gain and infinite input resistance), the resistors R_1 and R_2 completely define the small-signal gain of the non-inverting configuration. The following derivation indicates the mathematical gain formulae in the case where the op-amp has a finite internal gain and input resistance.

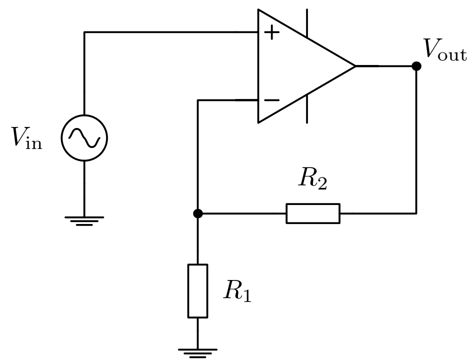

Figure 4: Non-inverting operational amplifier configuration

Prior to determining the output impedance of the non-inverting op-amp configuration it is necessary to calculate the voltage gain of the amplifier stage. Since voltages are the object of interest, the nodal analysis method is suitable for determining the relationship between the output and input voltages. The nodal analysis requires the signal sources to be represented as current sources. Due to this, the small-signal equivalent circuit needs to be redrawn after applying the required source transformations. Figure 2 shows the small-signal model, from where the voltage gain of the non-inverting op-amp configuration can be evaluated with nodal analysis.

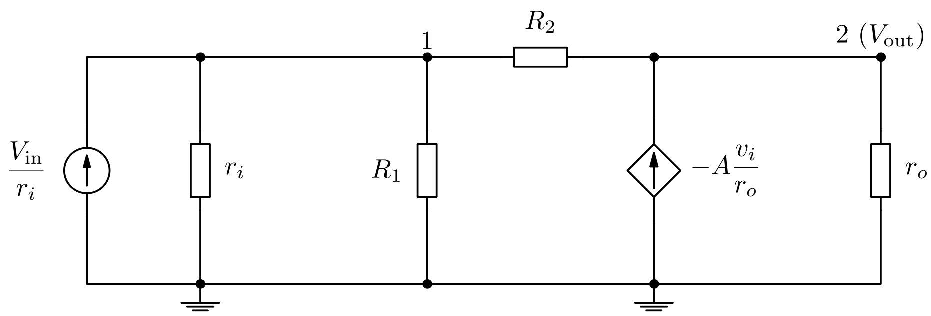

Figure 5: Small-signal model of the non-inverting operational amplifier

After the input voltage source and the voltage-controlled voltage source have been converted to current sources with the internal input resistance r_i and the internal output resistance r_o respectively, the small-signal model in Figure 5 is represented by the matrix equation

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{r_i} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{r_o} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{V_{\text{in}}}{r_i} \\ -\frac{Av_i}{r_o} \end{bmatrix}.$$

To ease out the calculation process, the gain term $-\frac{Av_i}{r_o}$ still needs to be transferred from the current vector to the admittance matrix. Because the small-signal input voltage $v_i = V_1 - V_{\text{in}}$ (verify this from Figure 6), the final form of the matrix equation is

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{r_i} & -\frac{1}{R_2} \\ -\frac{1}{R_2}+\frac{A}{r_o} & \frac{1}{R_2}+\frac{1}{r_o} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{V_{\text{in}}}{r_i} \\ \frac{V_{\text{in}}}{r_o} \end{bmatrix}.$$

Solving for the node voltage V_2, which is the output voltage V_{out}, yields a fractional gain term

$$V_{\text{out}} = V_2 = \frac{V_{\text{in}}\left(AR_1 + AR_2\right) + V_{\text{in}}R_1\frac{r_o}{r_i}}{R_1\left(1+A\right) + r_o + R_2 + \frac{r_oR_1}{r_i} + \frac{R_1R_2}{r_i}}.$$

This expression for the output voltage is used later together with the expression of the short circuit current when solving the output impedance of the non-inverting op-amp configuration.

Another small-signal model is needed to determine the short circuit current using the mesh analysis. Figure 6 indicates the chosen current loops and their directions inside the circuit branches.

Based on the small-signal model of Figure 6, the short circuit current is evaluated from the matrix equation

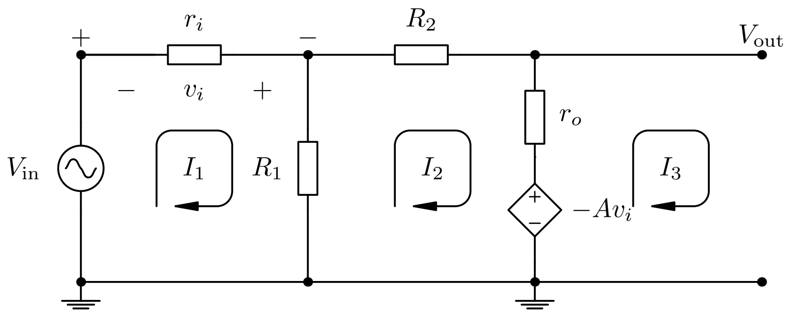

Figure 6: Small-signal model of the non-inverting operational amplifier

$$\begin{bmatrix} r_i + R_1 & -R_1 & 0 \\ -R_1 & R_1 + R_2 + r_o & -r_o \\ 0 & -r_o & r_o \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} \\ Av_i \\ -Av_i \end{bmatrix}.$$

The voltage v_i can be expressed using the mesh currents as $v_i = -I_1 r_i$, and the matrix can be reshaped to

$$\begin{bmatrix} r_i + R_1 & -R_1 & 0 \\ -R_1 + Ar_i & R_1 + R_2 + r_o & -r_o \\ -Ar_i & -r_o + AR_2 & r_o \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} \\ 0 \\ 0 \end{bmatrix}.$$

The solution of this matrix equation with respect to the mesh current I_3 is

$$I_3 = \frac{V_{\text{in}} \left(AR_1 + AR_2\right) + V_{\text{in}} R_1 \dfrac{r_o}{r_i}}{r_o \left(R_1 + R_2\right) + R_1 R_2 \dfrac{r_o}{r_i}}.$$

The output impedance is therefore

$$Z_{\text{out}} = \frac{V_2}{I_3} = \frac{r_o \left(R_1 + R_2\right) + R_1 R_2 \dfrac{r_o}{r_i}}{R_1 \left(1 + A\right) + r_o + R_2 + \dfrac{r_o R_1}{r_i} + \dfrac{R_1 R_2}{r_i}}.$$

This result for the output impedance of the non-inverting op-amp is exactly the same as for the inverting operational amplifier configuration.