Some random properties of the finite cosine series

October 29, 2012

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This little detour in the never-ending world of mathematics was inspired by the Fourier series and the Discrete Fourier Transform, both of which frequently bring up the following problem. The aim is to prove that the sum

$$\sum_{n=0}^{N-1} \cos\left(n\frac{2\pi}{N}\right), \tag{1}$$

where N can be any positive integer, equals 0. In the most general case one can write the finite series as

$$y = \cos(0) + \cos(x) + \cos(2x) + \cos(3x) + \ldots + \cos([N-1]x). \tag{2}$$

To shorten up this finite series into a more reasonable equation, both sides of (2) are multiplied with the term $2\sin x$:

$$2\sin(x)\cdot y = 2\cos(0)\sin(x) + 2\cos(x)\sin(x) + \ldots + 2\cos([N-1]x)\sin(x). \tag{3}$$

Now when applying the trigonometric identity

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta),$$

equation (3) transforms into:

$$\begin{aligned} 2\sin(x)\cdot y = 2\sin(x) & \\ &+ \sin(x+x) - \sin(x-x) \\ &+ \sin(2x+x) - \sin(2x-x) \\ &+ \sin(3x+x) - \sin(3x-x) \\ &+ \ldots \\ &+ \sin([N-1]x+x) - \sin([N-1]x-x) \end{aligned} \tag{4}$$

and after cleanup one is left with the following equation:

$$\begin{aligned} 2\sin(x)\cdot y &= \sin(x) + \sin([N-1]x) + \sin(Nx) \\ &= \sin(x) + \sin(-x) + \sin(2\pi) \\ &= \sin(x) - \sin(x). \end{aligned} \tag{5}$$

After dividing by the factor $2\sin(x)$ one has

$$y = \frac{1}{2} - \frac{1}{2} = 0. \tag{6}$$

This completes the 'proof' that the sum of all the terms in the cosine series always equals zero. Or, at least when there are more than three terms in the series.

On the other hand, from equation (1) one can distinguish two different cases of odd and even N. If one concentrates on the arguments inside the cosines, the odd N gives

$$0, \pi - \frac{\pi}{N}, \pi - \frac{3\pi}{N}, \ldots, \pi - \frac{(N-2)\pi}{N}, \pi + \frac{\pi}{N}, \pi + \frac{3\pi}{N}, \ldots, \pi + \frac{(N-2)\pi}{N}, \tag{7}$$

and the even N gives:

$$0, \pi - \frac{2\pi}{N}, \pi - \frac{4\pi}{N}, \ldots, \pi - \frac{(N-2)\pi}{N}, \pi, \pi + \frac{2\pi}{N}, \pi + \frac{4\pi}{N}, \ldots, \pi + \frac{(N-2)\pi}{N}, \tag{8}$$

and in the general case any phase shift ϕ can be added to these finite series to get

$$0, (\pi+\phi) - \frac{2\pi}{N}, (\pi+\phi) - \frac{4\pi}{N}, \ldots, (\pi+\phi) - \frac{(N-2)\pi}{N}, (\pi+\phi), \ldots \tag{9}$$

For both of these series one can apply the trigonometric identity

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

where the first cosine gets the constant value $\cos(\pi+\phi)$ and the second always the same finite series starting from 0 and ending to the argument value $\frac{(N-2)\pi}{N}$.

The series for the odd N has an extremely neat property. When the series is reduced completely inside the 1st quarter of the unit circle, one has

$$y = \cos(x) - \cos(2x) + \cos(3x) - \ldots \pm \cos(mx), \tag{10}$$

where $x = \frac{\pi}{N}$ and $m = \frac{(N-1)}{2}$.

Again, multiply equation (10) with $2\sin(x)$ to get:

$$2\sin(x)\cdot y = 2\cos(x)\sin(x) - 2\cos(2x)\sin(x) + \ldots \pm 2\cos(mx)\sin(x), \tag{11}$$

and use the identity

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta),$$

to have

$$\begin{aligned} 2\sin(x)\cdot y = {} & \sin(x+x) - \sin(x-x) \\ & - \sin(2x+x) + \sin(2x-x) \\ & + \sin(3x+x) - \sin(3x-x) \\ & + \ldots \\ & \pm \sin(mx+x) \mp \sin(mx-x) \end{aligned} \tag{12}$$

and after cleanup one is left with the following equation:

$$2\sin(x)\cdot y = \sin(x) \pm [\sin(mx) - \sin(mx+x)] \tag{13}$$

where the part $\sin(mx) - \sin(mx+x)$ yields

$$\sin(mx) - \sin(mx+x) = \sin\left(mx + \frac{x}{2} - \frac{x}{2}\right) - \sin\left(mx + \frac{x}{2} + \frac{x}{2}\right), \tag{14}$$

which equals

$$-2\sin\left(\frac{x}{2}\right)\cos\left(mx + \frac{x}{2}\right). \tag{15}$$

Therefore one has the equation

$$2\sin(x)\cdot y = \sin(x) \pm \left[-2\sin\left(\frac{x}{2}\right)\cos\left(mx + \frac{x}{2}\right)\right]. \tag{16}$$

Dividing by $2\sin(x)$ yields:

$$\begin{aligned} y &= \frac{1}{2} \pm \left[-\frac{2\sin\left(\frac{x}{2}\right)}{\sin(x)}\cos\left(mx + \frac{x}{2}\right)\right] \\ &= \frac{1}{2} \pm \left[-\frac{\cos\left(mx + \frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right], \end{aligned} \tag{17}$$

and after substituting the actual values for x and m:

$$\begin{aligned} y &= \frac{1}{2} \pm \left[-\frac{\cos\left[\left(\frac{N-1}{2}\right)\frac{\pi}{N} + \frac{1}{2}\frac{\pi}{N}\right]}{\cos\left(\frac{x}{2}\right)}\right] \\ &= \frac{1}{2} \pm \left[-\frac{\cos\left(\frac{\pi}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right] \\ &= \frac{1}{2}. \end{aligned} \tag{18}$$