Modelling a cassette tape rewinding process

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Firstly a word of warning. The following analysis and derivation are trivial and don't really give any help for practical re-use of cassette tapes in guitar effects or any other applications.

The background for this writing is that I have always wanted to derive a useful mathematical model for a mechanical system by first finding a proper differential equation and then solving it into an equation to get the state of the system in any point of time. Unfortunately this is not it. Here I just use a given formula, differentiate it, and integrate it back to the same form as it was earlier. Basically the same result could have been obtained directly by tweaking the initial simple formula to contain time dependence.

But maybe one day I will manage to come up with something very clever...

As a youngster I had always thought that the cassette tape must travel at a constant speed at all times. When I started thinking about it more carefully (several years of hard pondering ...) I came to the conclusion that the speed of the tape is never constant, only the angular speed of the reel is constant. But then I wanted to get my thougths officially confirmed just in case. After doing some 'serious' research (using www search engines) I had to throw all my hard pondering work to the trash bin. The tape speed is actually constant in play mode, where the capstan and pinch roller control the linear speed of the tape making it constant. So nothing interesting there then.

But the pinch roller is not pinching when rewinding the tape, so then I wanted to know how the tape speed changes during rewind. The capstan is still there rolling along, but most likely it does not affect much during rewind. I tested this assumption by rewinding a tape into an empty reel for 60 seconds then flipping the cassette and rewinding the same amount of tape back into the almost full reel. This took only 37 seconds, so the linear speed indeed depends on the radius of the reel in the rewind process. The following calculation analyses tape rewind mathematically.

The 'problem' should be made more clear using an illustration, which is given in Figure 1. Here the left reel turns clockwise at a constant angular velocity ω, and the linear speed of the tape v is measured from the linear section.

For this problem, basic physics offers the equation

$$v = \omega r = f 2\pi r, \tag{1}$$

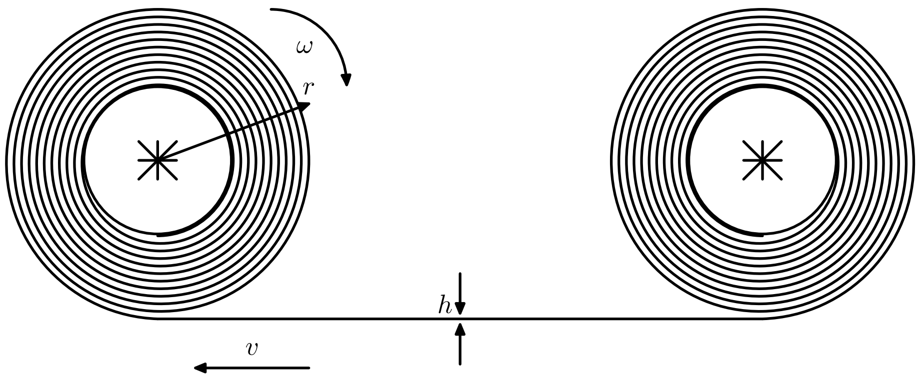

Figure 1: Tape inside a C-cassette (simplified sketch drawing)

where the linear velocity v is directly proportional to the product of angular velocity ω and the radius r of the rotating round object, which in this case consists of the cassette reel and the rolled tape on top of the reel. In addition, we know that $\omega = 2\pi f$, where f in this particular context designates the frequency of full revolutions of the reel. Clearly during one full revolution, the tape travels the distance $2\pi r$.

Now, as the time passes, more and more tape is packed on top of the reel and the radius is therefore dependent of time. Based on this, the following details on the time dependency can be added to our little equation:

$$v(t) = f2\pi r(t) \qquad ; \qquad v(0) = f2\pi r_0. \tag{2}$$

The initial velocity measured in the origin of time is when the radius has the constant value r_0. Or in other words, r_0 is the radius of the reel at time $t = 0$.

Next we want to know how the velocity changes with respect to time, so we operate with the differential operator on both sides of the equation. This is the result of the operation:

$$\frac{dv(t)}{dt}dt = f2\pi\frac{dr(t)}{dt}dt. \tag{3}$$

Now the brain is needed to figure out how to express the rate of change of the radius $r(t)$. After many nights without sleep, this came into mind:

$$\frac{dr(t)}{dt} = fh, \tag{4}$$

where f is still the same frequency of revolution and h designates the height (or thickness) of the cassette tape. Practically this means that with every full revolution the radius grows by the thickness of the tape. At least to my mind it makes sense. According to wikipedia, typical thickness of a cassette tape is between $10 - 16$ μm. These values can be used for obtaining numerical results from the resulting equation.

With the new expression for the rate of change of the radius, the differential equation reads

$$\dot{v}(t)dt = f^2 2\pi h \; dt. \tag{5}$$

Now we can solve the equation by integrating it:

$$\int \dot{v}(t)dt = f^2 2\pi h \int dt, \tag{6}$$

and the result of integration is

$$v(t) = f^2 2\pi h(t + C), \tag{7}$$

where C is the integration constant. To get the complete solution, the value of the integration constant needs to be determined. Since it is known that at time $t = 0$, $v = f2\pi r_0$, C can be solved from

$$v(0) = f2\pi r_0 = f^2 2\pi hC \qquad \Rightarrow \qquad C = \frac{r_0}{fh}. \tag{8}$$

And from here it follows that

$$v(t) = f2\pi r_0 + f^2 2\pi ht, \tag{9}$$

which is the final solution.

As a side note, it is also possible to include the initial value directly in the integration phase. All that is needed is to integrate from the initial time value to t, which is the same generic time variable (not a constant!) as already used inside the velocity function $v(t)$. Commonly here the integration differential variable dt is noted as $d\hat{t}$ or by some completely other letter and claimed it to be a 'dummy' variable. This is only possible in pure mathematics where the concept of differential is not recognised. In physics the differential represents a very small quantity with dimensions and therefore cannot be treated as a 'dummy' variable.

Anyway, the equation (6) can be written using the initial value in the integration limits as:

$$\int_{t=0}^{t} \dot{v}(t)dt = f^2 2\pi h \int_{t=0}^{t} dt. \tag{10}$$

When performing the simple integrations on both sides, the remaining equation reads:

$$v(t) - v(0) = f^2 2\pi h t - 0. \tag{11}$$

After inserting the initial value $v(0) = f 2\pi r_0$ into (11) one has the same solution as in (9):

$$v(t) = f 2\pi r_0 + f^2 2\pi h t. \tag{12}$$

Well, it is obvious that this is a linear relation with respect to time, so one cannot even draw any neat graphs based on this. Already from the basic equation (1) it is clear that when the radius doubles also the linear velocity of the tape doubles. The obtained equation can be used for estimating the tape speed at any given time after the tape started rolling at time $t = 0$. The initial reel radius r_0 can be chosen freely.